MAC-CPTM Situations Project

Situation 42: Sin (2x)

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Prompt

During a lesson on transformations of the sine function a student asks, "Why is the graph of $y = \sin(2x)$ a horizontal shrink of the graph of $y = \sin(x)$ instead of a horizontal stretch?"

Commentary

The function $y = \sin(kx)$, $k > 0$, transforms the horizontal length of the period of $y = \sin(x)$ by a factor of $1/k$. The first two foci emphasize the periodicity of the sine function, appealing to composition of functions and to the unit circle. In contrast, the third focus emphasizes the first derivative of $y = \sin(2x)$ as the instantaneous rate of change of the function.

Mathematical Foci

Mathematical Focus 1

The function $g(x) = \sin(2x)$ is a composition of the functions $f(x) = \sin(x)$ and $h(x) = ax$,a≠0. The graph of the composition has a period that reflects whether |a| *causes a horizontal shrink or stretch of the graph.*

The composition uses the outputs of $h(x) = 2x$ as inputs of $f(x) = \sin(x)$ [Figure 1]. Sin (x) has period 2π. Therefore, the period of $g(x) = \sin(2x)$ is π [Figure 2].

When a periodic function has a horizontal shrink transformation, the horizontal width of one cycle of the transformed function will be shorter than width of one cycle of the original function. Therefore, $g(x) = \sin(2x)$ has period π radians and each cycle of $f(x) = \sin(x)$ has horizontal width 2π radians, the graph of $g(x) = \sin(2x)$ will be a horizontal shrink of the graph of $f(x) = \sin(x)$.

x	sin(x)	2x	sin(2x)
0.000	**0.000**	**0.000**	**0.000**
0.262	0.259	**0.524**	**0.500**
0.524	**0.500**	**1.047**	**0.866**
0.785	0.707	**1.571**	**1.000**
1.047	**0.866**	**2.094**	**0.866**
1.309	0.966	**2.618**	**0.500**
1.571	**1.000**	**3.142**	**0.000**
1.833	0.966	**3.665**	**-0.500**
2.094	**0.866**	**4.189**	**-0.866**
2.356	0.707	**4.712**	**-1.000**
2.618	**0.500**	**5.236**	**-0.866**
2.880	0.259	**5.760**	**-0.500**
3.142	**0.000**	**6.283**	**0.000**
3.403	-0.259	6.807	0.500
3.665	**-0.500**	7.330	0.866
3.927	-0.707	7.854	1.000
4.189	**-0.866**	8.378	0.866
4.451	-0.966	8.901	0.500
4.712	**-1.000**	9.425	0.000
4.974	-0.966	9.948	-0.500
5.236	**-0.866**	10.472	-0.866
5.498	-0.707	10.996	-1.000
5.760	**-0.500**	11.519	-0.866
6.021	-0.259	12.043	-0.500
6.283	**0.000**	12.566	0.000

Figure 1

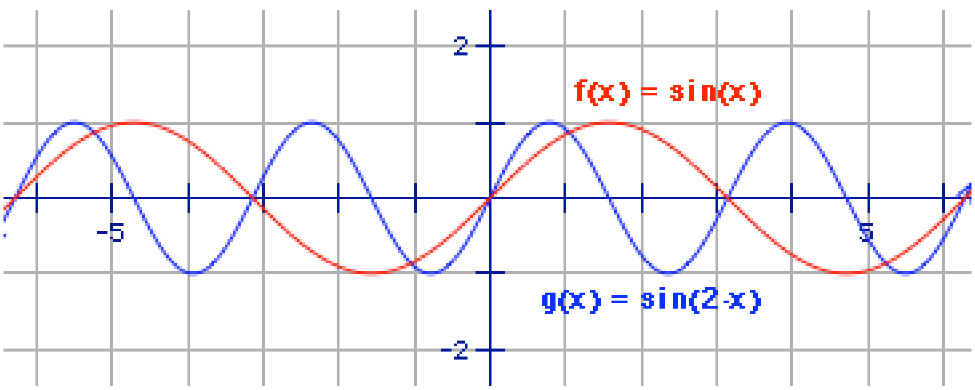

Figure 2

Mathematical Focus 2

The unit circle is the locus of all points with coordinates $(\cos(\theta),\sin(\theta))$. *The period of sin(θ) is 2π. The period of sin(2x) is the value of* x *that is needed for the values of 2x to run from 0 to 2π.*

Consider a right triangle with vertices at the origin, on the x-axis, and a point, A, on the unit circle with coordinates $(\cos(\theta),\sin(\theta))$ [Figure 3].

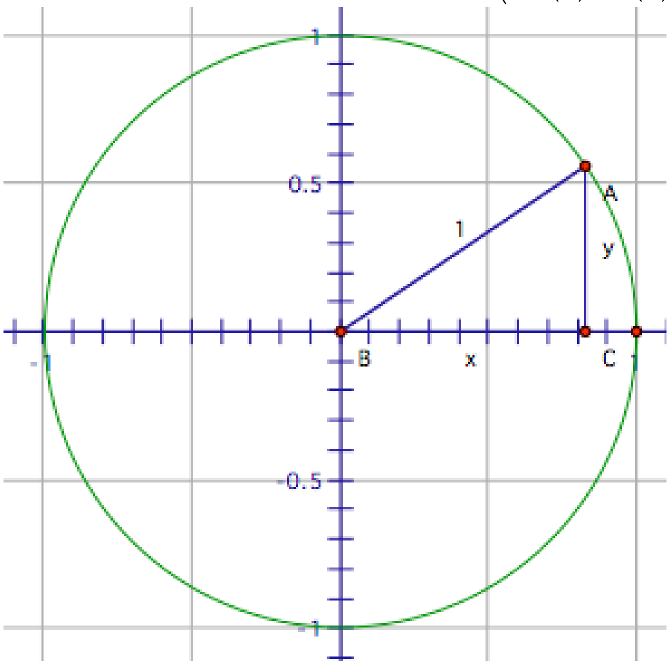

Figure 3

Transform the triangle by moving A around the unit circle. As A travels around the unit circle, the set of ordered pairs $(\cos(\theta),\sin(\theta))$, for the function $f(x)=\sin(x)$, is generated. One period of $f(x)=\sin(x)$ will be completed when A travels one full rotation around the unit circle, a distance of 2π radians. Hence, the horizontal length of one period of $f(x)=\sin(x)$ will be 2π radians. The graph of the function $f(x)=\sin(2x)$ can also be produced from the set of ordered pairs, $(\cos(2\theta),\sin(2\theta))$, generated by moving A around the unit circle. Since the input of the function $f(x)=\sin(2x)$ is $2x$, one period of $f(x)=\sin(2x)$ will be completed when A travels a distance of π radians. As a result, the period of $f(x)=\sin(2x)$ will be π radians, producing a horizontal shrink of the graph of $f(x)=\sin(x)$.

Mathematical Focus 3

The first derivative represents the instantaneous rate of change of a function and can be used in particular cases to locate maximum and minimum values of a function.

The derivative of y=sin(x) is y′ =cos(x), and the derivative of y=sin(2x) is y′ =2*cos(2x). Since the first derivative represents the instantaneous rate of change of a function, the graph of y=sin(2x) should oscillate between its maxima and minima at twice the rate of the graph of y=sin(x). Suppose that the graph of y=sin(x) were stretched horizontally, then the transformed graph should oscillate between its maxima and minima at a lesser rate than the graph of y=sin(x).

Post Commentary

The set of foci provide evidence that “multiplying by 2” in one circumstance might not mean “multiplying by 2” in a related circumstance. Sometimes the effect of multiplying by 2 in one setting relates to dividing by 2 in another setting (e.g., the relationship between periods of sin(x) and sin(2x)) or to neither multiplying or dividing (e.g., derivative of sin(2x) is twice the derivative of sin(x) but not twice sin(2x).